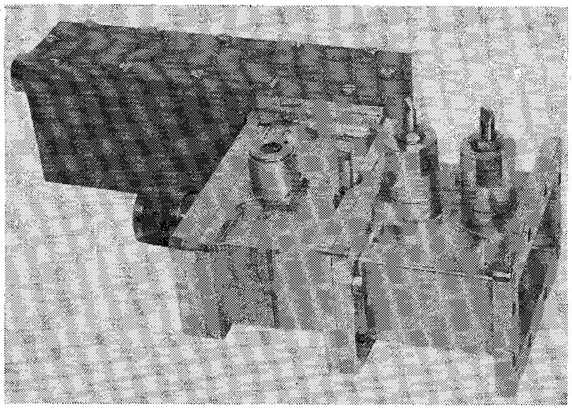

Fig. 5. Low-noise 4-GHz down-converter.

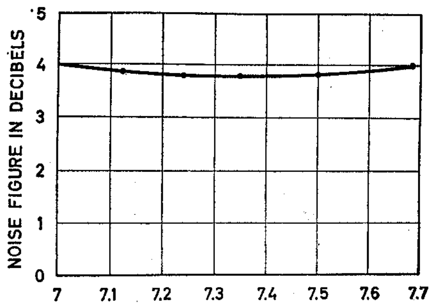

Fig. 6. Measured noise figure response of representative down-converter.

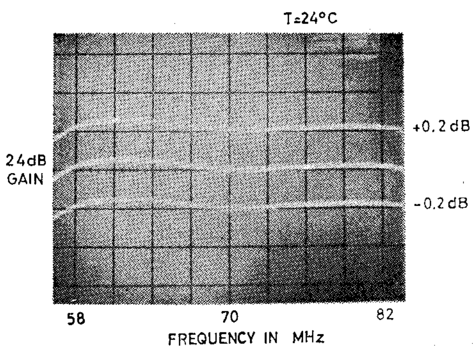

Fig. 7. Representative down-converter instantaneous overall gain-frequency response.

TABLE II

SUMMARY OF MEASURED PERFORMANCE OF RADIO-LINK MIXERS

RF input-signal tuning range	3.6–4.2 GHz	7.1–7.7 GHz	12.7–13.3 GHz
Overall noise figure at ambient temperature (including losses of image reject filter)	3.2 dB min 3.5 dB max	3.7 dB min 4 dB max	5 dB min 5.3 dB max
IF noise figure	1.5 dB	1.5 dB	1.5 dB
Conversion loss	2 dB max	2.6 dB max	4 dB max
Intermediate frequency		70 MHz	
Overall down-conversion gain of the receiver		24 dB	
Frequency response over any 25-MHz band	flat ±0.05 dB at ambient temperature ±0.2 dB from −5°C–55°C ±0.2 dB with ±2-dB pump power variation		
Diodes employed	HP 5082–2709	HP 5082–2709	AEI DC 1306
Rectified current	1 mA	1 mA	2 mA
AM/PM conversion	0.2°/dB at an input RF level of −20 dBm		

IV. Experimental Results

Down-converters in the configuration described in Section III have been realized in the whole frequency range from 4 to 13 GHz. The measured performance of three down-converter versions, designed for microwave radio-link equipment, operating in the frequency ranges of 3.6–4.2 GHz, 7.1–7.7 GHz, and 12.7–13.3 GHz, respectively, are summarized in Table II.

The RF tunable range, indicated in Table II, is covered by mechanically retuning the image rejection filter only. No tuning is needed for both the second-harmonic and third-harmonic idle-frequency filters. It has been found experimentally that the position of the image rejection filter is not critical. Only two electrical distances (i.e., only one spacer) are sufficient to cover the entire RF input-signal tuning range.

Typical behavior of the noise figure of a down-converter unit within the RF frequency range is shown in Fig. 6. In addition to the results given in Table II, a typical example of an instantaneous gain-frequency response is shown in Fig. 7; such a response is practically the same in the whole RF frequency range.

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Matching Considerations of Lossless Reciprocal 5-Port Waveguide Junctions

A. L. HIEBER AND R. J. VERNON

***Abstract*—Some of the restrictions imposed on general 5-port junctions (or networks) by losslessness and reciprocity are discussed as well as considerations of restrictions due to physical symmetry. It is proven that if a lossless reciprocal 5-port junction (or network) is completely matched, then all off-diagonal elements of the scattering matrix are nonzero; i.e., if the junction is matched, no port is decoupled from any of the others. It is also shown that all off-diagonal scattering coefficients of a lossless reciprocal 5-port junction (or network) have a magnitude of one half if and only if the junction is completely matched. Those physical symmetries which preclude complete matching of 5-port junctions are given and a general theorem concerning the matching of junctions and physical symmetry is proven.**

I. Introduction

It has long been established that lossless reciprocal waveguide junctions[1] having three or four electrical ports exhibit certain properties pertaining to complete matching and port decoupling, which may be determined from the fact that the scattering matrix of such a junction must be both unitary and symmetric [1], [6]. In addition,

Manuscript received July 7, 1972; revised January 8, 1973. This work was supported by the National Science Foundation through a traineeship for A. L. Hieber.
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[1] The more general term "network" may be used throughout this short paper in place of the terms "waveguide junction" except where the physical symmetry of the junction is specified.

for a waveguide junction possessing physical symmetry, group theory and matrix algebra may be employed to further determine the characteristics resulting from this symmetry [1]–[5]. The literature has considered in detail the general properties of lossless reciprocal 3- and 4-port junctions as well as many specific examples of each of these classes. Many specific junctions having five or more ports have also been discussed [1], [7]–[14] but general considerations of junctions having five or more ports have been virtually ignored. Such general considerations are often of great value in design procedures as has proven to be the case for 3- and 4-port junctions.

This short paper considers some restrictions imposed on general 5-port junctions by losslessness and reciprocity, and later some restrictions imposed by physical symmetry. It will be shown as follows that lossless reciprocal 5-port junctions cannot be completely matched if they have two or more ports decoupled from one another or if they have certain common forms of physical symmetry. Although these properties diminish the practical importance of 5-port junctions, the results themselves are none the less important since they specify quite general characteristics which cannot be physically achieved with the kind of structure being considered. It is also proven that for any lossless reciprocal 5-port junction which is completely matched, the off-diagonal elements of the scattering matrix all have a magnitude of one half, and relations between the phases of these elements are presented.

II. General 5-Port Junctions[2]

Consider a lossless reciprocal 5-port waveguide junction[3] (or network) which initially will also be assumed completely matched. Denote this junction by J and its scattering matrix by $[S]$. For a reciprocal junction, $[S]$ is symmetric so that $S_{ij}=S_{ji}$. Also, since the junction has been assumed to be completely matched, $S_{ii}=0$, $i=1,\cdots,5$. Employing these restrictions, the scattering matrix for J is given by

$$[S]=\begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{12} & 0 & S_{23} & S_{24} & S_{25} \\ S_{13} & S_{23} & 0 & S_{34} & S_{35} \\ S_{14} & S_{24} & S_{34} & 0 & S_{45} \\ S_{15} & S_{25} & S_{35} & S_{45} & 0 \end{bmatrix}. \tag{1}$$

Furthermore, since J is lossless, $[S]$ is unitary and the columns of $[S]$ form an orthonormal set under a Hermitian inner product, that is, $c_i{\dagger}\, c_j=\delta_{ij}$ where c_i is the ith column of $[S]$ written in column vector form, and $c_i{\dagger}=(c_i^*)^T$, where * denotes complex conjugation and T indicates the transpose operation. Applying this restriction to the columns of $[S]$ results in the following equations:

$$c_1{\dagger}c_1 = 1 = |S_{12}|^2+|S_{13}|^2+|S_{14}|^2+|S_{15}|^2 \tag{2}$$
$$c_2{\dagger}c_2 = 1 = |S_{12}|^2+|S_{23}|^2+|S_{24}|^2+|S_{25}|^2 \tag{3}$$
$$c_3{\dagger}c_3 = 1 = |S_{13}|^2+|S_{23}|^2+|S_{34}|^2+|S_{35}|^2 \tag{4}$$
$$c_4{\dagger}c_4 = 1 = |S_{14}|^2+|S_{24}|^2+|S_{34}|^2+|S_{45}|^2 \tag{5}$$
$$c_5{\dagger}c_5 = 1 = |S_{15}|^2+|S_{25}|^2+|S_{35}|^2+|S_{45}|^2 \tag{6}$$
$$c_1{\dagger}c_2 = 0 = S_{13}^*S_{23}+S_{14}^*S_{24}+S_{15}^*S_{25} \tag{7}$$
$$c_1{\dagger}c_3 = 0 = S_{12}^*S_{23}+S_{14}^*S_{34}+S_{15}^*S_{35} \tag{8}$$
$$c_1{\dagger}c_4 = 0 = S_{12}^*S_{24}+S_{13}^*S_{34}+S_{15}^*S_{45} \tag{9}$$
$$c_1{\dagger}c_5 = 0 = S_{12}^*S_{25}+S_{13}^*S_{35}+S_{14}^*S_{45} \tag{10}$$
$$c_2{\dagger}c_3 = 0 = S_{12}^*S_{13}+S_{24}^*S_{34}+S_{25}^*S_{35} \tag{11}$$
$$c_2{\dagger}c_4 = 0 = S_{12}^*S_{14}+S_{23}^*S_{34}+S_{25}^*S_{45} \tag{12}$$
$$c_2{\dagger}c_5 = 0 = S_{12}^*S_{15}+S_{23}^*S_{35}+S_{24}^*S_{45} \tag{13}$$
$$c_3{\dagger}c_4 = 0 = S_{13}^*S_{14}+S_{23}^*S_{24}+S_{35}^*S_{45} \tag{14}$$
$$c_3{\dagger}c_5 = 0 = S_{13}^*S_{15}+S_{23}^*S_{25}+S_{34}^*S_{45} \tag{15}$$
$$c_4{\dagger}c_5 = 0 = S_{14}^*S_{15}+S_{24}^*S_{25}+S_{34}^*S_{35}. \tag{16}$$

These equations will be used in the proof of Theorems 1 and 2 which follow.

Theorem 1

For a lossless reciprocal 5-port waveguide junction (or network), being completely matched and having two or more ports decoupled from one another are mutually exclusive conditions.

Proof: Since at least one lossless reciprocal 5-port junction can be completely matched (Dicke's star junction[4]), a proof by contradiction may be used by assuming that a lossless reciprocal 5-port junction J is completely matched and has at least two decoupled ports. Assuming both of these conditions simultaneously leads to a mathematical contradiction as is shown as follows.

Since it has been hypothesized that at least two ports of J are decoupled, let the ports be numbered so that at least ports 4 and 5 are decoupled. Then $S_{45}=0$.

Next it will be shown that S_{23} cannot equal zero if $S_{45}=0$ for the restrictions already placed on $[S]$. Assume that $S_{23}=0$. Then, (12)–(15) and $S_{45}=0$ imply that: 1) $S_{12}=S_{13}=0$, or 2) $S_{14}=S_{15}=0$, or 3) at least three of the four scattering coefficients S_{12}, S_{13}, S_{14}, and S_{15} are zero.

1) Setting $S_{12}=S_{13}=S_{23}=S_{45}=0$, in (2)–(6) and adding (5) and (6) yields

$$2=(|S_{14}|^2+|S_{15}|^2)+(|S_{24}|^2+|S_{25}|^2)+(|S_{34}|^2+|S_{35}|^2). \tag{17}$$

Now substituting (2)–(4) into (17) produces the obvious contradiction $2=3$. Thus, S_{12} and S_{13} cannot both be zero if $S_{23}=0$.

2) Duplicating the procedure in 1) for $S_{14}=S_{15}=S_{23}=S_{45}=0$ results in the same contradiction; hence, S_{14} and S_{15} cannot both be zero if $S_{23}=0$.

3) If at least three of the four coefficients S_{12}, S_{13}, S_{14}, and S_{15} are zero, then at least $S_{12}=S_{13}=0$ or $S_{14}=S_{15}=0$. But neither of these conditions can occur according to the developments under 1) and 2) if both $S_{45}=0$ and $S_{23}=0$.

Therefore, $S_{23}\neq 0$.

Noting that S_{45} has been assumed to be zero and that $S_{23}\neq 0$, (12)–(15) may be solved for S_{34}, S_{35}, S_{24}, and S_{25}, respectively. The expressions for S_{34} and S_{24} may be substituted into (9) giving

$$0=S_{12}^*S_{13}^*S_{14} \tag{18}$$

and the expressions for S_{35} and S_{25} may be substituted into (10) yielding

$$0=S_{12}^*S_{13}^*S_{15}. \tag{19}$$

Equations (18) and (19) have only the following possible solutions: 1) $S_{12}=0$, 2) $S_{13}=0$, 3) $S_{14}=S_{15}=0$, or 4) any combination of 1), 2), and 3).

1) If $S_{12}=0$, then since $S_{45}=0$, (12) and (13) indicate that $S_{34}=S_{35}=0$ also. Making these substitutions in (2)–(6) and adding (2) and (3) yields

$$2=(|S_{13}|^2+|S_{23}|^2)+(|S_{14}|^2+|S_{24}|^2)+(|S_{15}|^2+|S_{25}|^2). \tag{20}$$

Substituting (4)–(6) into (20) yields the result $2=3$, contradicting the hypothesis that $S_{12}=0$. Therefore, $S_{12}\neq 0$.

2) Duplication of the procedure in 1) for $S_{13}=0$ contradicts the hypothesis that $S_{13}=0$ is an acceptable solution to (18) and (19). Hence, $S_{13}\neq 0$.

3) If $S_{14}=S_{15}=0$, then $S_{24}=S_{25}=S_{34}=S_{35}=0$ also, from (12)–(15). When these six scattering coefficients vanish, (5) reduces to the contradiction, $1=0$. Thus, S_{14} and S_{15} cannot both be zero.

4) Any combination of the conditions of 1), 2), and 3) may similarly be shown to lead to contradictory results.

Therefore, $S_{45}\neq 0$ for a completely matched J.

In this proof the two ports assumed to be decoupled were arbitrarily numbered 4 and 5. However, port numbering can in no way influence the actual physical characteristics of a junction. Thus the same contradictory results would have been obtained if any pair of numbers had been assigned to the ports assumed to be decoupled. Also, in the above development all off-diagonal elements other than S_{45} were taken to be completely arbitrary unless it was proven that they must be nonzero. Thus the above development shows that it is *not* possible to have *any* combination of decoupled ports in a matched

[2] The term "general" is used here to indicate that the 5-port junctions considered in this section need not have any particular physical symmetry.
[3] All junctions (or networks) will also be considered linear and passive.

[4] It will be shown later that other possible symmetries also allow complete matching.

lossless reciprocal 5-port junction, and the proof of Theorem 1 is complete.

Theorem 2

All off-diagonal scattering coefficients of a lossless reciprocal 5-port junction (network) have a magnitude of one half if and only if the junction is completely matched.

Proof: Let J denote a lossless reciprocal 5-port network. The sufficient or "if" condition is proven first.

Since J is totally matched, then the scattering matrix is given by (1), and (2)–(16) apply. Also, by Theorem 1, $S_{ij} \neq 0$, $i \neq j$. Substituting the expression for S_{34} obtained from (9) into (8) yields

$$S_{35} = \frac{-S_{12}^*S_{13}^*S_{23} + S_{12}^*S_{14}^*S_{24} + S_{14}^*S_{15}^*S_{45}}{S_{13}^*S_{15}^*}. \tag{21}$$

Substituting this expression for S_{35} into (10) yields

$$S_{45} = \frac{S_{12}^*(S_{13}^*S_{23} - S_{14}^*S_{24} - S_{15}^*S_{25})}{2S_{14}^*S_{15}^*}. \tag{22}$$

Substituting this expression for S_{45} back into (21) gives

$$S_{35} = \frac{S_{12}^*(-S_{13}^*S_{23} + S_{14}^*S_{24} - S_{15}^*S_{25})}{2S_{13}^*S_{15}^*}. \tag{23}$$

Substituting the expression for S_{45} obtained from (22) into (9) gives

$$S_{34} = \frac{S_{12}^*(-S_{13}^*S_{23} - S_{14}^*S_{24} + S_{15}^*S_{25})}{2S_{13}^*S_{14}^*}. \tag{24}$$

Using (7), (22)–(24) become, respectively

$$S_{45} = \frac{S_{12}^*S_{13}^*}{S_{14}^*S_{15}^*}S_{23}, \quad S_{35} = \frac{S_{12}^*S_{14}^*}{S_{13}^*S_{15}^*}S_{24}, \quad S_{34} = \frac{S_{12}^*S_{15}^*}{S_{13}^*S_{14}^*}S_{25}. \tag{25}$$

Replacing (8)–(10) with (11)–(13), respectively, in the above procedure, and using the complex conjugate of (7), then the following relationships, analogous to (25), result:

$$S_{45} = \frac{S_{12}^*S_{23}^*}{S_{24}^*S_{25}^*}S_{13}, \quad S_{35} = \frac{S_{12}^*S_{24}^*}{S_{23}^*S_{25}^*}S_{14}, \quad S_{34} = \frac{S_{12}^*S_{25}^*}{S_{23}^*S_{24}^*}S_{15}. \tag{26}$$

In terms of scattering coefficient magnitudes, (25) and (26) reduce to the following:

$$|S_{1m}| = |S_{2m}|, \qquad m = 3, 4, 5. \tag{27}$$

But (27) must be valid independent of the port numbering of J, therefore, it must apply to the scattering coefficients relating any two ports of J to the remaining ports, i.e.

$$|S_{km}| = |S_{lm}|, \quad m \neq k, l, \quad k, l, m = 1, \cdots, 5. \tag{28}$$

Substituting (28) into (2)–(6) and remembering that $|S_{ij}| = |S_{ji}|$, $i, j = 1, \cdots, 5$, then

$$|S_{ij}| = \tfrac{1}{2}, \qquad i \neq j, \qquad i, j = 1, \cdots, 5 \tag{29}$$

and the sufficient or "if" condition is proved.

Now, consider the necessary or "only if" condition. If all off-diagonal scattering coefficients of J have a magnitude of one half, i.e., $|S_{ij}| = \frac{1}{2}$, $i \neq j$, $i, j = 1, \cdots, 5$, then since $[S]$ is unitary

$$c_l^\dagger c_l = 1 = \sum_{k=1}^{5} |S_{kl}|^2 = 1 + |S_{ll}|^2 \tag{30}$$

and

$$0 = S_{ll}, \qquad l = 1, \cdots, 5. \tag{31}$$

Equation (31) is synonymous to the specification that J is completely matched; hence, the necessary condition is proven.

Theorem 2 specifies all scattering coefficient magnitudes for a lossless reciprocal completely matched 5-port junction. If (7)–(16), or equivalently (14)–(16), (25), and (26) are solved in terms of the scattering coefficient phases, five equations in ten unknowns result. This exhibits the extent to which these phases may be specified for a matched reciprocal lossless 5-port junction. The results are listed in Table I for independently chosen ϕ_{12}, ϕ_{13}, ϕ_{14}, ϕ_{15}, and ϕ_{23}, where $S_{ik} = |S_{ik}| \exp(j\phi_{ik})$. Of course, if the junction under consideration possesses physical symmetry, more information than is given in the table may be attainable as shown by Dicke for the case of the 5-port star junction [1].

TABLE I

SCATTERING COEFFICIENT PHASE RELATIONSHIPS FOR A LOSSLESS RECIPROCAL COMPLETELY MATCHED 5-PORT JUNCTION

$\phi_{24} = -\phi_{13} + \phi_{14} + \phi_{23} + 2\pi(n_1 \pm 1/3)$
$\phi_{25} = -\phi_{13} + \phi_{15} + \phi_{23} + 2\pi(n_2 \mp 1/3)$
$\phi_{34} = -\phi_{12} + \phi_{14} + \phi_{23} + 2\pi(n_3 \mp 1/3)$
$\phi_{35} = -\phi_{12} + \phi_{15} + \phi_{23} + 2\pi(n_4 \pm 1/3)$
$\phi_{45} = -\phi_{12} - \phi_{13} + \phi_{14} + \phi_{15} + \phi_{23} + 2\pi n_5$

Note: Phase relationships are for independently chosen ϕ_{12}, ϕ_{13}, ϕ_{14}, ϕ_{15}, and ϕ_{23}. Here $S_{ik} = \frac{1}{2}\exp(j\phi_{ik})$, $i \neq k$ and n_k, $k = 1, \ldots, 5$, is an integer. The upper signs give one acceptable set of phase relations and the lower signs give another.

III. SYMMETRICAL 5-PORT WAVEGUIDE JUNCTIONS

In this section 5-port waveguide junctions possessing some form of physical symmetry are considered. An object possesses physical symmetry if one or more operations such as reflection in a plane, or rotation (other than by 360°) about an axis superimpose the object back into itself [1]. These operations which leave the object unchanged are called covering or symmetry operations. The set of all such operations for a physical object, together with the identity operation, form a group in the mathematical sense [15]. If a covering operation of a waveguide junction is performed just on the fields within the junction, keeping the same junction position and port numbering, the new field pattern will also satisfy the boundary conditions within the junction, but the fields at the various ports will in general be redistributed. The matrix which describes this redistribution of the port electric fields by a covering operation is termed the port electric-field transformation for that operation. The port electric-field transformations for any given junction also form a group which is isomorphic with the group of covering operations for that junction. These port-field transformations must commute with the scattering matrix so that [1]

$$[X][S] = [S][X] \tag{32}$$

where $[X]$ denotes the port-field transformation associated with a covering operation X.

If $G = \{[X_1], \cdots, [X_k]\}$ is a group of port-field transformations of a junction J and if every covering operation of J has a corresponding port-field transformation which is an element of G, then G is said to completely represent the symmetry exhibited by J. Furthermore, if for all $[X]$ in G, the restrictions imposed by (32) on the scattering coefficients do not preclude the possibility that $S_{ii} = 0$, $i = 1, \cdots, n$, then it is said that the symmetry of J does not preclude J from being completely matched. Similarly, if for all $[X]$ in G, (32) does require that $S_{ii} \neq 0$ for one or more values of i, then it is said that the junction's symmetry precludes its being completely matched. Note that if symmetry does not preclude complete matching, there is still no guarantee that J can necessarily be matched. For example, copper plates located at the terminal planes of all the ports of J will preserve the symmetry of J and yet leave the junction so that it cannot be matched.

Theorem 3

Let J_1 and J_2 be two lossless reciprocal n-port junctions, and let G and H be groups of port-field transformations, completely representing the symmetry of J_1 and J_2, respectively. If symmetry does not preclude totally matching J_1 and if H is a subgroup of G, then symmetry does not preclude totally matching J_2.

Proof: Since H is a subgroup of G, then all port-field transformations of H are contained in G. Thus the restrictions placed on the scattering coefficients of J_2 by H and (32) are satisfied if the restrictions placed on the scattering coefficients of J_1 by G and (32) are satisfied. But, symmetry does not preclude totally matching J_1; thus the scattering coefficient restrictions may be satisfied for a completely matched J_1 and also for a totally matched J_2. Therefore, symmetry does not preclude completely matching J_2 as was to be shown.

TABLE II

MATCHING PROPERTIES OF LOSSLESS RECIPROCAL 5-PORT JUNCTIONS COMPOSED OF RECTANGULAR, CIRCULAR, AND/OR COAXIAL WAVEGUIDES AND POSSESSING VARIOUS FORMS OF PHYSICAL SYMMETRY

Junction Symmetry	Example	Precludes Complete Matching
Fifth-order rotational*	Fig. 1	No
Fourth-order rotational*	Fig. 2	Yes
Third-order rotational*	Fig. 3	Yes
Second-order rotational		
Type 2	Fig. 4	No
Type 1	Fig. 5	Yes
Type 0**	—	—
First-order rotational†		No
Reflection		
Type 2	Fig. 4	No
Type 1	Fig. 6	Yes
Type 0	Fig. 7	No††

* The results are valid for a junction composed of arbitrary waveguides.

** This symmetry is unrealizable in the five-port case.

† First-order rotational symmetry, generated by the identity operation, never precludes matching any junction.

†† This result is valid if and only if the representative port-field transformation is $\pm [I_5]$.

The converse of Theorem 3 does not hold in general. If G represents symmetry that precludes matching J_1 and if $H=\{[I_n]\}$,[5] where I denotes the identity operation and $[I_n]$ denotes the $n\times n$ identity matrix which is the port-field transformation for I, then H is a subgroup of G. But (32) is always satisfied for $[X]=[I_n]$. Therefore, symmetry does not preclude matching J_2. This situation is a counterexample of the converse of Theorem 3, and the converse is invalid. Other less trivial counterexamples also exist.

Theorem 3 is quite useful in conjunction with a symmetry class analysis of lossless reciprocal n-port junctions, i.e., a determination of the types of physical symmetry that preclude complete matching and those which do not. This evaluation is most easily performed by considering cyclic groups of symmetry operations and their associated port-field transformations. Since these groups are generated by one group element, then all scattering coefficient restrictions imposed by a cyclic group may be ascertained by commuting only this group element with the scattering matrix in accordance with (32). If it can be shown that such a cyclic group precludes complete matching, e.g., the scattering coefficient restrictions imposed by (32) are not consistent with the relationships of Table I for the 5-port case, then any group containing this cyclic group will also preclude matching by Theorem 3. Since every mathematical group may be subdivided into cyclic subgroups, then any group of port-field transformations which precludes total matching may be determined from the cyclic groups which preclude complete matching, providing all possible cyclic groups are considered.

Such a symmetry class analysis of lossless reciprocal 5-port waveguide junctions has been performed [16]. While the analysis itself is too lengthy to include here, the results are given in Theorems 4 and 5, and Table II. For purposes of conveniently classifying symmetry operations, the following definitions are introduced.

Definition 1

If a waveguide junction J has a rotational symmetry operation about an axis A by $2\pi m/n$ rad, where m and n are relatively prime integers, then there exist $n-1$ distinct additional rotations about A [15], and J is said to have nth-order rotational symmetry. The axis A is said to be an n-fold axis.

Note that a 5-port junction can have at most five distinct rotations about one axis, hence it can have rotational symmetry of at most the fifth order about one axis.

Definition 2

If an n-port waveguide junction J has a reflection operation about one plane P or a second-order rotation operation about one axis A and if the corresponding port-field transformation spatially affects the fields[6] in $2k$ of the junctions ports, $2k\leq n$, then J is said to have reflection symmetry or second-order rotational symmetry, respectively, of type k.

Note that a 5-port junction can have at most four ports whose fields are spatially affected by a reflection or second-order rotation. Thus it can have reflection or second-order rotational symmetry of at most type 2. Figs. 1–7 illustrate the notation and terminology introduced in these definitions.

With this terminology in mind, the following theorems on 5-port junctions are introduced.

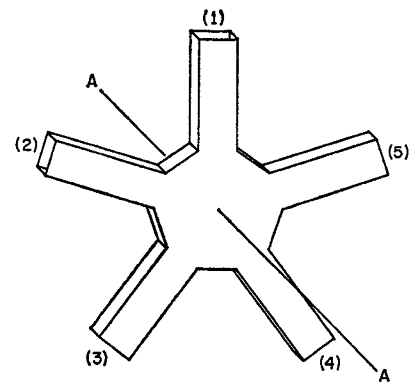

Fig. 1. Typical 5-port junction possessing fifth-order rotational symmetry about an axis A.

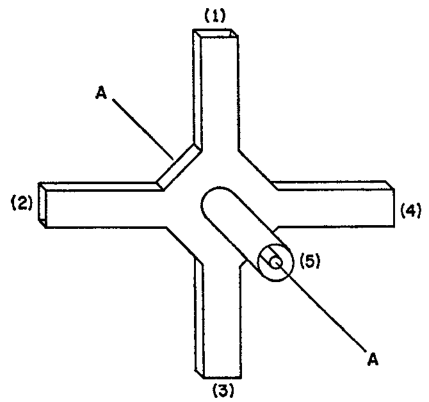

Fig. 2. Typical 5-port junction possessing fourth-order rotational symmetry about an axis A.

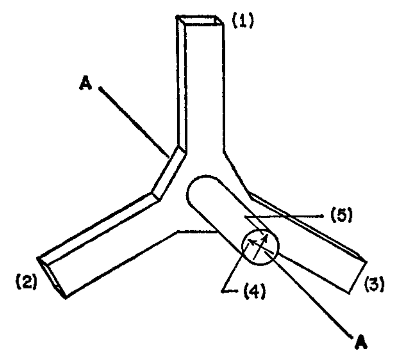

Fig. 3. Typical 5-port junction possessing third-order rotational symmetry about an axis A.

[5] Since H completely represents the symmetry of J_2, then this is equivalent to specifying that J_2 has no symmetry. Consequently, symmetry does not preclude matching J_2.

[6] If only the sense of the fields is affected by a port-field transformation, then the fields are not considered to be spatially affected.

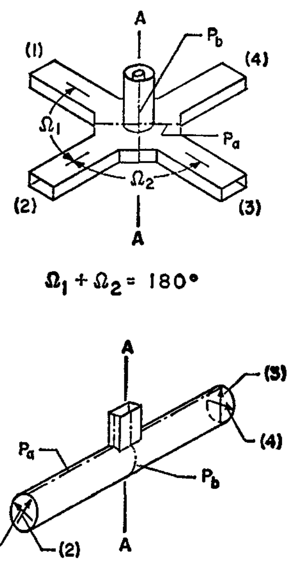

Fig. 4. Typical 5-port junctions possessing type 2 second-order rotational symmetry about an axis A and type 2 reflection symmetry about each of the planes P_a and P_b.

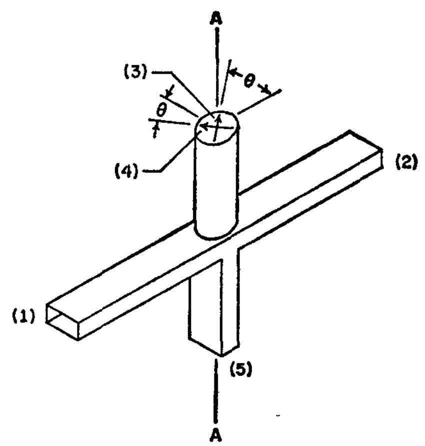

Fig. 5. Typical 5-port junction possessing type 1 second-order rotational symmetry about an axis A.

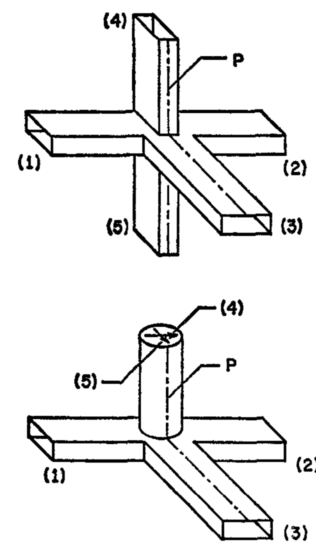

Fig. 6. Typical 5-port junctions possessing type 1 reflection symmetry about a plane P.

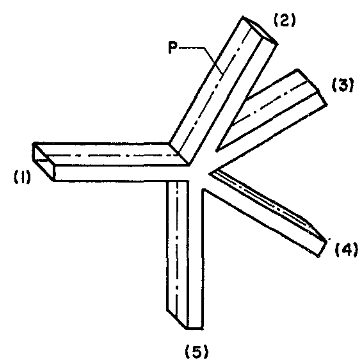

Fig. 7. Typical 5-port junction possessing type 0 reflection symmetry about a plane P.

Theorem 4

If a lossless reciprocal 5-port junction J has a group of port-field transformations G, completely representing the symmetry of J, and if G has a subgroup H which represents either: 1) a third-order rotational class of symmetry operations, or 2) a fourth-order rotational class of symmetry operations, then G precludes completely matching J.

Theorem 5

If a lossless reciprocal 5-port junction J composed of rectangular, circular, and/or coaxial waveguides has a group of port-field transformations G, completely representing the symmetry of J, and if G has a subgroup H which represents either: 1) a type 1 second-order rotational class of symmetry operations, or 2) a type 1 reflection class of symmetry operations, then G precludes completely matching J.

The matching properties of all cyclic symmetry groups that a 5-port can have are summarized in Table II. Finally, the theorem following gives further information concerning junctions having a reflection symmetry plane which cuts two or more ports such that the electric field of one is odd and the electric field of another is even, relative to that plane.

Theorem 6

If a lossless reciprocal 5-port waveguide junction has a plane of reflection symmetry P which cuts two or more electrical ports so that the electric field of one is even and that of another is odd with respect to P, then the junction may not be matched (and still retain the reflection symmetry).

Proof: By the "port decoupling theorem" [5], if the fields of two ports of a reciprocal junction have opposite symmetry (one even and one odd) with respect to a plane of reflection symmetry of the junction, they are decoupled. Then, by Theorem 1, since the junction being considered has five electrical ports, it cannot be matched and the theorem is proven.

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Faster Impedance Estimation for Coupled Microstrips with an Overrelaxation Method

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***Abstract*—Using the Frankel-Young method [1], [2], fast estimation of the potential distribution for a microstrip structure is obtained when an accelerating factor ω is introduced in the finite-differences (relaxation) method. It is possible to calculate such a factor by an iterative technique, but the time of computation needed to find ω annihilates the theoretical gain.**

In this short paper, the authors present a method which gives an analytical expression for ω. The realistic case examined here, as an illustration, is that of the suspended microstrip couplers for which odd and even impedances are the interesting parameters.

Given an analytical expression for ω, the overrelaxation method appears as a very powerful and attractive method for finding the solution of any type of microstrip structure.

I. Introduction

The integrated technology using microstrips provides new possibilities for microwave designs. A very important one is the realization of compact low-cost dispersive lines used as group-velocity correctors for digital telecommunications. The basic component of such a system can be reduced to a microstrip coupler.

In the last few years, several authors [3]–[7] have treated some particular problems using different methods, but they are generally complicated and applicable to particular geometrical cases.

A solution using finite differences has been proposed by Green [8] and others. An application has been given by Brenner [9] to the simple case of the suspended microstrip line and by Gupta [10] to the idealized problem considered by Cohn [3], i.e., the suspended coupler in a homogeneous dielectric such as air. That problem is purely theoretical, with no substrate sustaining the strips.

However, as emphasized by Smith [6], the methods using finite differences appear as inadequate because the very fine mesh required for the accuracy leads to difficulties in the convergence. Clearly, it means that the computing time becomes prohibitive and the computer memory becomes saturated.

Nevertheless, the use of the finite-differences method should become a very fruitful approach if an accelerating factor taking into account the geometry of the problem could be injected in the program.

In this short paper, the problem of the research of such a factor is solved and applied to the case of the suspended coupler with a dielectric substrate of constant ϵ_r sustaining the strips.

In the finite-differences method, we define in a geometrical domain the potentials at the nodes of a net (Fig. 1). The relations between all the potentials can be written

$$(A)(\Psi) = (B). \tag{1}$$

System (1) can be solved by an iterative process [1], [2], [11] writing

$$\Psi_{i+1} = M\Psi_i + C. \tag{2}$$

The Frankel–Young method introduces the accelerating factor ω. An optimal value of $\omega - \omega_{\text{opt}}$ gives the fastest convergence. We have

$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \lambda_M{}^2}} \tag{3}$$

Manuscript received July 10, 1972; revised March 16, 1973. This work was supported by the Centre National d'Etudes des Télécommunications, Lannion, France.
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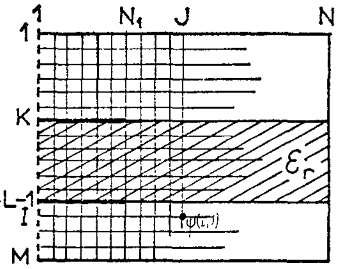

Fig. 1. Geometrical parameters definition.

$$\lambda_M = \sup \left| 1 - \frac{\mu_k}{a_{kk}} \right| \tag{4}$$

where μ_k are the eigenvalues and a_{kk} are the diagonal elements of A.

II. Applications of the Frankel–Young Method to a Suspended Microstrip Coupler

A. Resolution of the Problem for an "Empty Box" [11]

Using the finite-differences method and for the second-order approximation, the Laplace's equation is reduced to

$$\Psi(I, J) = \tfrac{1}{4}[\Psi(I, J-1) + \Psi(I, J+1) + \Psi(I-1, J) + \Psi(I+1, J)]. \tag{5}$$

The variable changing $\psi(I, J) = X_i$, with $i = (N-2)(I-2) + J - 1$, allows us to have the unknowns indexed by a continuous sequence (Fig. 1).

The ith equation of the system (1) will be written

$$4X_i - X_{i-1} - X_{i+1} - X_{i+N-2} - X_{i-N+2} = 0, \qquad 1 \le i \le (M-2)(N-2).$$

The matrix A_1 (Fig. 2) can always be split into two symmetrical tridiagonals—matrices A_1 and A_2, the main diagonal elements being $a_{kk}/2$ such that $A = A_1 + A_2$.

It can be shown that A_1 and A_2 have the same eigenvectors. Let V be one of these eigenvectors and μ_1 and μ_2 be the two corresponding eigenvalues for A_1 and A_2; thus we have

$$AV = (\mu_1 + \mu_2)V.$$

Consequently, the eigenvalue of A corresponding to V is

$$\mu = \mu_1 + \mu_2.$$

The matrix A_1 has $(M-2)$ tridiagonal blocks of order $(N-2)$, where all are identical; let A_1' be such a block. The eigenvalues μ_1 of A_1 are $(M-2)$ times the eigenvalues of A_1':

$$A_1' = \left| \begin{array}{cccccc} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{array} \right| N-2$$

The eigenvalues of the matrix A_1' are

$$\mu_{1k}' = 2 - 2\cos\frac{k\Pi}{N-1}, \qquad 1 \le k \le N-2.$$

In the matrix A_2, by permutations of rows and columns, it is possible to reduce A_2 to a band matrix $\overline{A}_2$ like A_1.

In order to avoid the tedious calculations by permutations, it is possible to find a faster process to transform A_2 into $\overline{A}_2$. One chooses another variable changing $\psi(I, J) = X_i{}^*$, with $i = (M-2)(J-2) + I - 1$.

The system (1) is then written $(A)^*(X)^* = (B)^*$. The physical problem is unchanged, so that the solution is the same. The solution vector $(X)^*$ is simply written on a new set of coordinate vectors. A and A^* represent the same linear application; consequently, they are similar and therefore have the same eigenvalues.

This time, A^* can be split into two matrices $A_1{}^*$ and $A_2{}^*$, where $A_1{}^*$ originates from the vertical lines and $A_2{}^*$ from the horizontal lines and $A^* = A_1{}^* + A_2{}^*$.

$A_1{}^*$ has $(N-2)$ diagonal blocks $A_1'{}^*$ similar to A_1', but of order $M-2$.